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Eigenvalue estimates for the Dirac–Schrödinger operators

Bertrand Morel

Institut Élie Cartan, Université Henri Poincaré, Nancy I, BP 239, 54506 Vandœuvre-Lès-Nancy Cedex, France

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Abstract

We give new estimates for the eigenvalues of the hypersurface Dirac operator in terms of the intrinsic energy–momentum tensor, the mean curvature and the scalar curvature. We also discuss their limiting cases as well as the limiting cases of the estimates obtained by Zhang and Hijazi [Math. Res. Lett. 5 (1998) 199; 6 (1999) 465; Ann. Glob. Anal. Geom., in press]. We compare these limiting cases with those corresponding to the Friedrich and Hijazi inequalities. We conclude by comparing these results to intrinsic estimates for the Dirac–Schrödinger operator $D_f = D - \frac{1}{2}f$.
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1. Introduction

In this paper, we start by comparing the hypersurface spinor bundle S of a hypersurface M to the fundamental spinor bundle ΣM of M. The hypersurface spinor bundle S is obtained by restricting the spinor bundle of the ambient space N to M. If $\varphi \in \Gamma(S)$ is a section of this bundle, the energy–momentum tensor Q^φ associated with φ is defined on the complement of its zero set, by

$$Q^\varphi_{ij} = \tfrac{1}{2}(e_i \cdot \nu \cdot \nabla_j \varphi + e_j \cdot \nu \cdot \nabla_i \varphi, \varphi/|\varphi|^2),$$

where ν is a unit normal vector field globally defined along M, e_i, e_j are vectors of a local orthonormal frame of M, and where $\nabla_i \varphi$ stands for the covariant derivative of the spinor field φ in the direction of e_i. Then the Schrödinger–Lichnerowicz formula for the classical Dirac operator D on M leads to the following result (compare with [12,13]):

E-mail address: morel@iecn.u-nancy.fr (B. Morel).

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Theorem 1.1. *Let $M^n \subset (N^{n+1}, \widetilde{g})$ be a compact hypersurface of a Riemannian spin manifold N. Let λ be any eigenvalue of the hypersurface Dirac operator $D_H = D - \frac{1}{2}H$, associated with an eigenspinor φ. Assume that $R + 4|Q^\varphi|^2 > H^2 > 0$, then one has*

$$\lambda^2 \geq \frac{1}{4}\inf_M \left(\sqrt{R + 4|Q^\varphi|^2} - |H|\right)^2, \tag{1}$$

where R and H are, respectively, the scalar curvature and the mean curvature of M, and Q^φ is the energy–momentum tensor associated with φ.

In fact, we see that if M is a minimal hypersurface, the hypersurface Dirac operator corresponds to the classical Dirac operator. Therefore, in this case, this estimate is exactly the one given by Hijazi in [8].

We then discuss the limiting case of Eq. (1) and that given by Zhang in [12,13].

As in [7,9], we then prove

Theorem 1.2. *Under the same conditions as in Theorem 1.1, suppose that $\bar{R}\,\mathrm{e}^{2u} + 4|Q^\varphi|^2 > H^2 > 0$, where $\bar{R}$ is the scalar curvature of M for some conformal metric $\bar{g} = \mathrm{e}^{2u}\widetilde{g}$, with $\mathrm{d}u(\nu)_{|M} \equiv 0$, then*

$$\lambda^2 \geq \frac{1}{4}\inf_M \left(\sqrt{\bar{R}\,\mathrm{e}^{2u} + 4|Q^\varphi|^2} - |H|\right)^2. \tag{2}$$

The discussion of the limiting case in this inequality and that proved in [9] is similar to that of (1). As a conclusion, we observe that these inequalities correspond to a generalization of the classical estimates in terms of the Dirac–Schrödinger operator $D_f = D - \frac{1}{2}f$, for a real function f on M.[1]

2. Preliminaries

2.1. Restriction of spinors to the hypersurface

In this paper we will consider an oriented compact hypersurface (M^n, g) of a Riemannian spin manifold $(N^{n+1}, \widetilde{g})$, with a spin structure Spin N. The metric g is the induced metric on M by $\widetilde{g}$. The possibility to define globally a unit normal vector field ν on M allows to induce from Spin N a spin structure on M, denoted by Spin M. For this, we can associate to every oriented orthonormal frame $(e_1, \ldots, e_n)$ on M an oriented orthonormal frame $(e_1, \ldots, e_n, \nu)$ of N such that the principal SO(n)-bundle $\mathrm{SO}_n M$ of oriented orthonormal frames on M is identified with a sub-bundle of $\mathrm{SO}_{n+1} N_{|M}$. Such a map is denoted by Φ.

Let $\mathbb{C}l_n$ be the n-dimensional complex Clifford algebra and $\mathbb{C}l_n^0$ its even part. Recall that there exists an isomorphism

[1] We would like to thank Oussama Hijazi for pointing out this problem, as well as Nicolas Ginoux and Xiao Zhang for helpful discussions.

$$\begin{array}{llll} \alpha: & \mathbb{C}l_n & \to & \mathbb{C}l^0_{n+1}, \\ & e_i & \mapsto & e_i \cdot \nu. \end{array} \tag{3}$$

Here, ν stands for the last vector of the canonical basis of $\mathbb{R}^{n+1}$.

In particular, α yields the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}(n) & \xrightarrow{\alpha} & \mathrm{Spin}(n+1) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \mathrm{Ad}} \\ \mathrm{SO}(n) & \hookrightarrow & \mathrm{SO}(n+1) \end{array},$$

where the inclusion of $\mathrm{SO}(n)$ in $\mathrm{SO}(n+1)$ is that which fixes the last basis vector under the action of $\mathrm{SO}(n+1)$ on $\mathbb{R}^{n+1}$, and Ad the adjoint representation of $\mathrm{Spin}(n)$ on $\mathrm{SO}(n)$, which is given by

$$\mathrm{Ad}_\eta(x) = \eta \cdot x \cdot \eta^{-1}$$

for all $\eta \in \mathrm{Spin}(n)$ and $x \in \mathbb{R}^n$.

This allows to pull back via Φ the fiber bundle $\mathrm{Spin}\, N_{|M}$ on $\mathrm{SO}\, M$ as a spin structure for M, denoted by $\mathrm{Spin}\, M$. The projection of $\mathrm{Spin}\, M$ on $\mathrm{SO}\, M$, as well as the projection of $\mathrm{Spin}\, N$ on $\mathrm{SO}\, N$, is denoted as π. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin} M & \xrightarrow{\Phi^*} & \mathrm{Spin} N_{|M} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathrm{SO} M & \overset{\Phi}{\hookrightarrow} & \mathrm{SO} N_{|M} \end{array},$$

Let ΣN be the spinor bundle on N, i.e.

$$\Sigma N = \mathrm{Spin}\, N \times_{\rho_{n+1}} \Sigma_{n+1},$$

where ρ_{n+1} is the restriction to $\mathrm{Spin}(n+1)$ of an irreducible representation of $\mathbb{C}l_{n+1}$ on the space of spinors Σ_{n+1}, of dimension $2^{[(n+1)/2]}$ ([.] denotes the integer part). Recall that if $n+1$ is odd, this representation is chosen so that the complex volume form acts as the identity on Σ_{n+1}.

Locally, by definition of ΣN, if U is an open subset of N and $\psi \in \Gamma_U(\Sigma N)$ a local section of the spinor bundle, we can write

$$\psi = [\widetilde{s}, \sigma],$$

where $\sigma : U \to \Sigma_{n+1}$ and $\widetilde{s} : U \to \mathrm{Spin}\, N$ are smooth maps, and $[\widetilde{s}, \sigma]$ is the equivalence class with respect to the relation

$$[\widetilde{s}, \sigma] \sim [\widetilde{s} g, \rho_{n+1}(g^{-1})\sigma], \quad \forall g \in \mathrm{Spin}(n+1).$$

Moreover, we can always assume that $\pi(\widetilde{s})$ is a local section of $\mathrm{SO}\, N$ with ν for last basis vector. Then we have

$$\psi_{|U\cap M} = [(\widetilde{s}_{|U\cap M}, \sigma_{|U\cap M})],$$

where the equivalence class is reduced to elements of $\mathrm{Spin}(n)$.

It follows that one can realize the restriction to M of the spinor bundle ΣN as

$$S := \Sigma N_{|M} = \mathrm{Spin}\, M \times_{\rho_{n+1}\circ\alpha} \Sigma_{n+1}.$$

Remark. The inclusion of $\mathrm{Spin}(n)$ in $\mathrm{Spin}(n+1)$ given by α is the trivial one. But, this notation emphasizes that Clifford multiplication of a spinor field $\phi \in \Gamma(S)$ by a vector X tangent to M is given by

$$(X, \phi) \mapsto X \cdot \nu \cdot \phi. \tag{4}$$

This fact is crucial for the following identification (see [1,2]).

2.2. *Identification of S with ΣM*

We now compare S with the intrinsic spinor bundle of M,

$$\Sigma M = \mathrm{Spin}\, M \times_{\rho_n} \Sigma_n.$$

For this, we have to examine the cases where n is even or odd. First assume that $n = 2m$ is even. From (3) and

$$\mathbb{C}l_{2m} \cong \mathbb{C}(2^m), \tag{5}$$

it follows that the representation of $\mathbb{C}l_{2m}$ given by $\rho_{2m+1}\circ\alpha$ is simply the restriction of ρ_{2m+1} to $\mathbb{C}l^0_{2m+1}$. But this representation is irreducible (see [10]). The representation $\rho_{2m+1}\circ\alpha$ is then an irreducible representation of $\mathbb{C}l_{2m}$ of dimension $\dim \Sigma_{2m+1} = 2^{[(2m+1)/2]} = 2^m$, as ρ_{2m}. Now, (5) implies that such a representation is unique up to an isomorphism. So $\rho_{2m} \cong \rho_{2m+1}\circ\alpha$ and we can conclude that

$$S \cong \Sigma M. \tag{6}$$

Let $\omega_{2m} = \mathrm{i}^m e_1 \ldots e_{2m}$ be the complex volume form in even dimension. An easy calculation shows that $\alpha(\omega) = \omega$. The decomposition of ΣM into positive and negative parts is preserved under the isomorphism (6) and we have

$$S = S^+ \oplus S^-,$$

where

$$S^{\pm} = \{\psi \in S | \mathrm{i}\nu\cdot\psi = \pm\psi\} \cong \Sigma M^{\pm}.$$

Indeed, because we choose ρ_{2m+1} as the irreducible representation of $\mathbb{C}l_{2m+1}$ for which the complex volume form $\omega_{2m+1} = \mathrm{i}^{m+1} e_1 \cdot\cdots\cdot e_{2m}\cdot\nu$ acts as the identity on Σ_{2m+1}, one has, for $\psi \in S$

$$\mathrm{i}\nu\cdot\psi = \mathrm{i}\nu\cdot\omega_{2m+1}\cdot\psi = \mathrm{i}^m\mathrm{i}^2\nu\cdot e_1\cdot\cdots\cdot e_{2m}\cdot\nu\cdot\psi = \omega_{2m}\cdot\psi.$$

Assume now that $n = 2m+1$ is odd. Recall the following isomorphism:

$$\mathbb{C}l_{2m+1} = \mathbb{C}(2^m) \oplus \mathbb{C}(2^m). \tag{7}$$

As mentioned above, ρ_{2m+1} corresponds to the irreducible representation of $\mathbb{C}l_{2m+1}$ for which the action of the complex volume form ω_{2m+1} is the identity. Because $n+1 = 2m+2$ is even, ΣN decomposes into positive and negative parts,

$$\Sigma N^{\pm} = \mathrm{Spin}\, N \times_{\rho^{\pm}_{2m+2}} \Sigma^{\pm}_{2m+2}.$$

If e_k is a basis vector tangent to M, then

$$\begin{aligned}\alpha(e_k)\cdot\omega_{2m+2} &= \mathrm{i}^{m+1} e_k\cdot\nu\cdot e_1\cdot\cdots\cdot e_{2m+1}\cdot\nu\\ &= \mathrm{i}^{m+1}(-1)^{2m+2}(-1)^{2m+2} e_1\cdot\cdots\cdot e_{2m+1}\cdot\nu\cdot e_k\cdot\nu = \omega_{2m+2}\,\alpha(e_k).\end{aligned}$$

So $\rho_{2m+2}\circ\alpha$ preserves the decomposition of ΣN, and

$$S = S^{+}\oplus S^{-},$$

with

$$S^{\pm} = \mathrm{Spin}\, M \times_{\rho^{\pm}_{2m+2}\circ\alpha} \Sigma^{\pm}_{2m+2},$$

and where ω_{2m+2} acts as $\pm\mathrm{Id}$ on $S^{\pm}$.

Moreover,

$$\alpha(\omega_{2m+1}) = \mathrm{i}^{m+1}(e_1\cdot\nu)\cdot\cdots\cdot(e_{2m+1}\cdot\nu) = \mathrm{i}^{m+1} e_1\cdot\cdots\cdot e_{2m+1}\cdot\nu = \omega_{2m+2},$$

and then ρ_{2m+1} and $\rho^{+}_{2m+2}\circ\alpha$ are both irreducible representations of $\mathbb{C}l_{2m+1}$ of the same dimension, such that $\rho_{2m+1}(\omega_{2m+1})$ and $\rho^{+}_{2m+2}\circ\alpha(\omega_{2m+1})$ are, respectively, the identity on Σ_{2m+1} and Σ^{+}_{2m+2}. Because such a representation is unique up to an isomorphism, we deduce that $\rho_{2m+1}\cong\rho^{+}_{2m+2}\circ\alpha$ and

$$S^{+}\cong\Sigma M. \tag{8}$$

Thus we have shown the following proposition.

Proposition 2.1. *If n is even (resp. odd), there exists an identification of the hypersurface spinor bundle S (resp. S^{+}) with the spinor bundle ΣM which sends every spinor $\varphi\in S$ (resp. S^{+}) to the spinor denoted by $\varphi^{*}\in\Sigma M$. Moreover, with respect to this identification, Clifford multiplication by a vector field X, tangent to M, is given by*

$$X\cdot\varphi^{*} = (X\cdot\nu\cdot\varphi)^{*}.$$

2.3. *The spinorial Gauss formula and the hypersurface Dirac operator*

Let $\widetilde{\nabla}$ be the Levi–Civita connection of $(N^{n+1}, \widetilde{g})$, and ∇ that of (M^n, g). Let $(e_1, \ldots, e_n, e_{n+1} = \nu)$ be a local orthonormal basis for *TM*, then the Gauss formula says that for $1\le i, j\le n$,

$$\widetilde{\nabla}_i e_j = \nabla_i e_j + h_{ij}\nu, \tag{9}$$

where h_{ij} are the coefficients of the second fundamental form of the hypersurface M. We are going to relate the associated connections on the corresponding spinor bundles. For this, consider $\phi \in \Gamma(\Sigma N)$ and $\varphi = \phi_{|M} \in \Gamma(S)$ its restriction to M. Recall now that locally, for $X \in \Gamma(TM)$,

$$\widetilde{\nabla}_X\phi = X(\phi) + \frac{1}{2}\sum_{1\le i<j\le n+1} \widetilde{g}(\widetilde{\nabla}_X e_i, e_j)e_i \cdot e_j \cdot \phi, \tag{10}$$

and

$$\begin{aligned}\nabla_X\varphi &= X(\varphi) + \frac{1}{2}\sum_{1\le i<j\le n} g(\nabla_X e_i, e_j)e_i \cdot \nu \cdot e_j \cdot \nu \cdot \varphi \\ &= X(\varphi) + \frac{1}{2}\sum_{1\le i<j\le n} g(\nabla_X e_i, e_j)e_i \cdot e_j \cdot \varphi.\end{aligned}$$

Therefore, by restricting both sides of Eq. (10) to M, and using the fact that $X(\phi)_{|M} = X(\phi_{|M})$ for X tangent to M, the Gauss formula (9) yields, for $1 \le k \le n$,

$$\begin{aligned}(\widetilde{\nabla}_k\phi)_{|M} &= e_k(\varphi) + \frac{1}{2}\sum_{1\le i<j\le n} \widetilde{g}(\nabla_k e_i + h_{ki}\nu, e_j)e_i \cdot e_j \cdot \varphi \\ &\quad + \frac{1}{2}\sum_{1\le i\le n} \widetilde{g}(\nabla_k e_i + h_{ki}\nu, \nu)e_i \cdot \nu \cdot \varphi \\ &= e_k(\varphi) + \frac{1}{2}\sum_{1\le i<j\le n} g(\nabla_k e_i, e_j)e_i \cdot e_j \cdot \varphi + \frac{1}{2}\sum_{1\le i\le n} h_{ki}e_i \cdot \nu \cdot \varphi \\ &= \nabla_k\varphi + \frac{1}{2}\sum_{1\le i\le n} h_{ki}e_i \cdot \nu \cdot \varphi.\end{aligned}$$

Once again, from Eq. (10), it makes sense to write $(\widetilde{\nabla}_X\phi)_{|M} = \widetilde{\nabla}_X\varphi$ when X is tangent to M, and hence we proved the spinorial Gauss formula

$$\forall\varphi \in \Gamma(S),\ \ \forall X \in \Gamma(TM), \quad \widetilde{\nabla}_X\varphi = \nabla_X\varphi + \tfrac{1}{2}h(X) \cdot \nu \cdot \varphi. \tag{11}$$

(Here h is seen as an endomorphism of the tangent bundle.)

It is known (see [10]) that there exists a positive definite Hermitian metric $\langle\cdot,\cdot\rangle$ on ΣN such that, if τ is a k-form on N,

$$\langle\tau\cdot\phi, \psi\rangle = (-1)^{k(k+1)/2}\langle\phi, \tau\cdot\psi\rangle, \quad \forall\phi, \psi \in \Gamma(\Sigma N). \tag{12}$$

If we denote $(\cdot,\cdot)$ its real part, we have

$$(\widetilde{X}\cdot\phi, \widetilde{Y}\cdot\phi) = \widetilde{g}(\widetilde{X}, \widetilde{Y})(\phi, \phi), \qquad (\widetilde{X}\cdot\phi, \phi) = 0, \quad \forall\widetilde{X}, \widetilde{Y} \in \Gamma(TN). \tag{13}$$

We simply restrict $(\cdot,\cdot)$ to M to get a globally defined metric on S. Now, because $\widetilde{\nabla}$ is compatible with $(\cdot,\cdot)$, i.e.

$$X(\varphi, \psi) = (\widetilde{\nabla}_X\varphi, \psi) + (\varphi, \widetilde{\nabla}_X\psi), \quad \forall\varphi, \psi \in \Gamma(S),\ \ \forall X \in \Gamma(TM).$$

Formula (11) easily implies that ∇ is also compatible with the metric. We remark that Eq. (11) implies that with respect to the identification of Proposition 2.1, we have

$$(\nabla\phi)^* = \nabla\phi^*. \tag{14}$$

This leads to the metric $(\cdot,\cdot)_{\Sigma M}$ on the intrinsic spinor bundle, with the same properties as $(\cdot,\cdot)$, and hence the two bundles are isometric.

Because Clifford multiplication of a spinor by a vector tangent to M is given by (4), if n is odd, S^+ is stable by ∇ and by Clifford multiplication. So the classical Dirac operator is simply defined on S for n even (resp. S^+ for n odd) by

$$D=\sum_{i=1}^{n} e_i\cdot\nu\cdot\nabla_i.$$

Now we define the hypersurface Dirac operator on $\Gamma(S)$ as

$$D_H=\sum_{i=1}^{n} e_i\cdot\nu\cdot\widetilde{\nabla}_i.$$

This definition is motivated by the following fact. Let

$$\widetilde{D}=\sum_{i=1}^{n} e_i\cdot\widetilde{\nabla}_i$$

be the hypersurface Dirac operator defined by Witten (see [6,11]) to prove the positive energy conjecture in general relativity. Then $\widetilde{D}$ is not formally self-adjoint with respect to the metric $(\cdot,\cdot)$. Indeed, it is proved in [9] that

$$D_H^2=\widetilde{D}^*\widetilde{D},$$

where $\widetilde{D}^*$ is the formal adjoint of $\widetilde{D}$ w.r.t. $(\cdot,\cdot)$.

From formula (11), we see that for n even (resp. odd), we have the following relations on $\Gamma(S)$ (resp. $\Gamma(S^+)$):

$$\begin{aligned}D_H&=\sum_i e_i\cdot\nu\cdot\nabla_i+\sum_i e_i\cdot\nu\cdot\frac{h(e_i)}{2}\cdot\nu\\&=D+\sum_{i,j}\frac{h_{ij}}{2}e_i\cdot e_j=D+\sum_{i,j}\frac{h_{ij}}{4}(e_i\cdot e_j+e_j\cdot e_i)=D-\sum_{i,j}\frac{h_{ij}}{2}\delta_{ij},\end{aligned}$$

and hence, if $H=\sum_i h_{ii}$ is the mean curvature of the hypersurface, we have

$$D_H=D-\frac{H}{2}. \tag{15}$$

In the following, we will not distinguish the cases where n is even or odd. In fact, if n is odd, D_H preserves the decomposition of S into positive and negative spinors, as well as Clifford multiplication (recall (4)), $\widetilde{\nabla}$ and ∇. Indeed, if $\phi\in\Gamma(S)$ is an eigenspinor of D_H with eigenvalue λ, it is the same for ϕ^+, its positive part. So we only consider positive spinors. The notation becomes easier with this convention.

Now, it is easy to see from Eq. (15) that D_H is formally self-adjoint with respect to the metric $(\cdot,\cdot)$ (see [9]). Finally, recall the well-known Schrödinger–Lichnerowicz formula on $\Gamma(\Sigma M)$ which by the previous identification is also true on $\Gamma(S)$:

$$D^2=\nabla^*\nabla+\tfrac{1}{4}R, \tag{16}$$

R being the scalar curvature of M and ∇^* the formal adjoint of ∇ with respect to the metric $(\cdot,\cdot)$.

3. Proof of the Theorem 1.1

Now we give an estimate for the eigenvalues of D_H in terms of the energy–momentum tensor (see [8]). For any spinor field $\varphi \in \Gamma(S)$, we define the associated energy–momentum 2-tensor Q^φ on the complement of its zero set, by

$$Q^\varphi_{ij} = \tfrac{1}{2}(e_i \cdot \nu \cdot \nabla_j\varphi + e_j \cdot \nu \cdot \nabla_i\varphi, \varphi/|\varphi|^2). \tag{17}$$

Remark 3.1. This definition corresponds to the one given in [8] if we note that with respect to the identification of S with ΣM of Proposition 2.1,

$$Q^\varphi_{ij} = \tfrac{1}{2}(e_i \cdot \nabla_j\varphi^* + e_j \cdot \nabla_i\varphi^*, \varphi^*/|\varphi^*|^2)_{\Sigma M}.$$

If φ is an eigenspinor for D_H, Q^φ is well defined in the sense of distribution. For any real functions p and q, consider the modified covariant derivative defined on S by

$$\nabla^Q_i = \nabla_i + \left(p\frac{H}{2} + q\lambda\right) e_i \cdot \nu + \sum_j Q^\varphi_{ij} e_j \cdot \nu. \tag{18}$$

Remark 3.2. This connection is well defined on S^+ when n is odd.

Using (13), we have

$$\begin{aligned}
|\nabla^Q\varphi|^2 &= |\nabla\varphi|^2 + n\left(p\frac{H}{2}+q\lambda\right)^2|\varphi|^2 + \sum_{i,j,k} Q^\varphi_{ij}Q^\varphi_{ik}(e_j\cdot\nu\cdot\varphi, e_k\cdot\nu\cdot\varphi)\\
&\quad +2\left(p\frac{H}{2}+q\lambda\right)\sum_i(\nabla_i\varphi, e_i\cdot\nu\cdot\varphi) + 2\sum_{i,j}Q^\varphi_{ij}(\nabla_i\varphi, e_j\cdot\nu\cdot\varphi)\\
&\quad +2\left(p\frac{H}{2}+q\lambda\right)\sum_{i,j}Q^\varphi_{ij}(e_i\cdot\nu\cdot\varphi, e_j\cdot\nu\cdot\varphi)\\
&= |\nabla\varphi|^2 + n\left(p\frac{H}{2}+q\lambda\right)^2|\varphi|^2 + |Q^\varphi|^2|\varphi|^2 - 2\left(p\frac{H}{2}+q\lambda\right)(D\varphi,\varphi)\\
&\quad -2|Q^\varphi|^2|\varphi|^2 + 2\left(p\frac{H}{2}+q\lambda\right)\mathrm{Tr}(Q^\varphi)|\varphi|^2,
\end{aligned}$$

but

$$\mathrm{Tr}(Q^\varphi)|\varphi|^2 = (D\varphi,\varphi),$$

hence

$$|\nabla^Q\varphi|^2 = |\nabla\varphi|^2 + n(p(\tfrac{1}{2}H)+q\lambda)^2|\varphi|^2 - |Q^\varphi|^2|\varphi|^2. \tag{19}$$

Now, since $D_H = D - \frac{1}{2}H$, the Schrödinger–Lichnerowicz formula (16) on $\Gamma(S)$ gives

$$\int_M |\nabla\varphi|^2 v_g = \int_M \left(|D\varphi|^2 - \frac{R}{4}|\varphi|^2\right) v_g = \int_M \left(\left(\lambda + \frac{H}{2}\right)^2 - \frac{R}{4}\right) |\varphi|^2 v_g. \tag{20}$$

Therefore (19) and (20) imply

$$\int_M |\nabla^Q\varphi|^2 v_g = \int_M \left((1+nq^2)\lambda^2 - \frac{R}{4} - |Q^\varphi|^2\right)|\varphi|^2 v_g + \int_M \left((1+np^2)\frac{H^2}{4} + (1+npq)H\lambda\right)|\varphi|^2 v_g. \tag{21}$$

Now, assume that q has no zeros so that we can choose $p = -1/nq$. Then (21) becomes

$$\int_M |\nabla^Q\varphi|^2 v_g = \int_M (1+nq^2)\left[\lambda^2 - \frac{1}{4}\left(\frac{R+4|Q^\varphi|^2}{1+nq^2} - \frac{H^2}{nq^2}\right)\right]|\varphi|^2 v_g. \tag{22}$$

If $R + 4|Q^\varphi|^2 > H^2 > 0$, we can take

$$nq^2 = \frac{|H|}{\sqrt{R+4|Q^\varphi|^2} - |H|}. \tag{23}$$

Then Eq. (22) becomes

$$\int_M |\nabla^Q\varphi|^2 v_g = \int_M (1+nq^2)[\lambda^2 - \tfrac{1}{4}(\sqrt{R+4|Q^\varphi|^2} - |H|)^2]|\varphi|^2 v_g. \tag{24}$$

Because the left-hand side of this equation is positive and λ is a constant, we get

$$\lambda^2 \geq \tfrac{1}{4}\inf_M(\sqrt{R+4|Q^\varphi|^2} - |H|)^2. \tag{25}$$

Remark 3.3. If M is minimal, i.e. $H = 0$, we can choose $q \equiv 0$ in (18) so that (25) specializes to the inequality of Theorem A in [8].

Remark 3.4. Note that our definition of the energy–momentum tensor Q^φ coincides with that in [8]. The definition used in [9,12,13] gives a factor $n/(n-1)$ in front of $R + 4|Q^\varphi|^2$ in inequality (25) but in this case, Q^φ has no canonical intrinsic meaning.

4. Limiting cases

First recall the inequality proved by Zhang.

Theorem 4.1 (Zhang [12,-16905]). *Let $M^n \subset N^{n+1}$ be a compact hypersurface of a Riemannian spin manifold $(N, \widetilde{g})$. Assume that $n \geq 2$ and $nR > (n-1)H^2 > 0$. Then if λ is any eigenvalue of the hypersurface Dirac operator D_H, one has*

$$\lambda^2 \geq \frac{1}{4}\inf_M\left(\sqrt{\frac{n}{n-1}R} - |H|\right)^2. \tag{26}$$

As in the proof of Theorem 1.1, the proof of Theorem 4.1 is based on the use of the modified connection

$$\nabla_i^\lambda = \nabla_i + (p(\tfrac{1}{2}H) + q\lambda)e_i \cdot \nu. \tag{27}$$

Here, p and q are related by

$$p = \frac{1-q}{1-nq}, \tag{28}$$

and

$$q = \frac{1}{n}\left(1 - \sqrt{\frac{(n-1)|H|}{\sqrt{n/(n-1)R} - |H|}}\right) \tag{29}$$

or, in other terms,

$$(1-nq)^2 = \frac{(n-1)|H|}{\sqrt{n/(n-1)R} - |H|}. \tag{30}$$

Equality holds in (26) for an eigenspinor φ of D_H with eigenvalue λ if and only if $\sqrt{n/(n-1)R} - |H|$ is constant and $\nabla^\lambda\varphi \equiv 0$. But, with respect to the identification of Proposition 2.1, and by (14), $\nabla^\lambda\varphi \equiv 0$ is equivalent to

$$\forall i = 1, \ldots, n, \quad \nabla_i\varphi^* = -(p(\tfrac{1}{2}H) + q\lambda)e_i \cdot \varphi^*. \tag{31}$$

It is known (see [7]) that if such a section exists on ΣM, then $p(H/2)+q\lambda$ has to be constant (say λ_1/n for instance) and that in this case M is Einstein and $R = 4((n-1)/n)\lambda_1^2$. So φ is a Killing spinor and we are in the limiting case of Friedrich's inequality [3]. Moreover, since $\sqrt{n/(n-1)R} - |H|$ is constant, H has to be constant.

Therefore, since $D\varphi = \lambda_1\varphi$ and $\lambda_1 = \frac{1}{2}\mathrm{sign}(\lambda_1)\sqrt{n/(n-1)R}$, the following equation must be satisfied (recall that $D_H = D - \frac{1}{2}H$):

$$\lambda = \frac{\mathrm{sign}(\lambda_1)}{2}\sqrt{\frac{n}{n-1}R} - \frac{H}{2} = \frac{\mathrm{sign}(\lambda_1)}{2}\sqrt{\frac{n}{n-1}R} - \mathrm{sign}(H)\frac{|H|}{2}. \tag{32}$$

But equality case gives

$$\lambda = \frac{\mathrm{sign}(\lambda)}{2}\left(\sqrt{\frac{n}{n-1}R} - |H|\right), \tag{33}$$

So (32) and (33) imply that

$$\mathrm{sign}(\lambda) = \mathrm{sign}(\lambda_1) = \mathrm{sign}(H). \tag{34}$$

On the other hand, an easy calculation leads to

$$\begin{aligned} p\frac{H}{2} + q\lambda &= \frac{\mathrm{sign}(\lambda)}{2n}\sqrt{\frac{n}{n-1}R} + \frac{(\mathrm{sign}(H) - \mathrm{sign}(\lambda))}{2n} \\ &\quad \times \left(1 + \sqrt{(n-1)\left(\sqrt{\frac{n}{n-1}R} - |H|\right)}\right) = \frac{\mathrm{sign}(\lambda_1)}{2n}\sqrt{\frac{n}{n-1}R} \end{aligned}$$

and we recover the already known fact that $p(H/2) + q\lambda = \lambda_1/n$.

Indeed, (34) can be trivially observed because in the equality case, both R and H are constant, so we can think of the spectrum of D_H as the shifting of the spectrum of D by $-\frac{1}{2}H$. Then the condition $nR > (n-1)H^2 > 0$ in Theorem 4.1 simply implies that the lowest eigenvalue of D_H (in the sense of its absolute value) must have the sign of H. In particular, when n is even, it shows how we lose the symmetry of the spectrum when passing from D to D_H (compare with the case where $H = 0$).

Now we discuss the case of Theorem 1.1. The limiting case of inequality (1) holds for an eigenspinor φ of D_H with eigenvalue λ if and only if $\nabla^Q\varphi \equiv 0$. First note that this implies that $|\varphi|$ is constant. Then, with respect to the identification of Proposition 2.1, and by (14), $\nabla^Q\varphi \equiv 0$ is equivalent to

$$\nabla_i\varphi^* = -\left(p\,\frac{H}{2} + q\lambda\right)e_i\cdot\varphi^* - \sum_j Q^\varphi_{ij}e_j\cdot\varphi^*. \tag{35}$$

Let $f = p(H/2) + q\lambda$, then Eq. (35) can be written as

$$\nabla_i\varphi^* = -\sum_j (Q^\varphi_{ij} + f\delta_{ij})\,e_j\cdot\varphi^*. \tag{36}$$

Now let $T_{ij} = Q^\varphi_{ij} + f\delta_{ij}$, taking Clifford multiplication by e_k on both sides of Eq. (36), yields

$$e_k\cdot\nabla_i\varphi^* = -\sum_j T_{ij}e_k\cdot e_j\cdot\varphi^*,$$

which gives

$$(e_k\cdot\nabla_i\varphi^*, \varphi^*)_{\Sigma M} = -\sum_j T_{ij}(e_k\cdot e_j\cdot\varphi^*, \varphi^*)_{\Sigma M},$$

and, because $(e_k\cdot e_j\cdot\varphi^*, \varphi^*)_{\Sigma M} = 0$ unless $j = k$ and T_{ij} is symmetric, we proved

$$\tfrac{1}{2}(e_i\cdot\nabla_k\varphi^* + e_k\cdot\nabla_i\varphi^*, \varphi^*/|\varphi^*|^2)_{\Sigma M} = T_{ik}.$$

Hence

$$T_{ik} = Q^\varphi_{ik},$$

and we can conclude that $f = 0$. Eq. (35) reduces to

$$\nabla_i\varphi^* = -\sum_j Q^\varphi_{ij}e_j\cdot\varphi^*. \tag{37}$$

Such field equations have been studied, as well as their integrability conditions, by Kim and Friedrich in [5]. Note that they allow a nice formulation of the theory of immersed surfaces in the Euclidean 3-space (see [3]). We will call an EM-spinor (for energy–momentum spinor) a non trivial spinor field satisfying (37). If it is an eigenspinor for the Dirac operator, which is equivalent to the fact that tr Q^φ is constant, it is called T-Killing spinor (see [4]). In fact, a T-Killing spinor is exactly a spinor field satisfying the limiting case in Hijazi's inequality [8].

Now we have (see [8] or Lemma 4.1(iii) of [5])

$$(\mathrm{tr}\, Q^{\varphi})^2 = \tfrac{1}{4}R + |Q^{\varphi}|^2. \tag{38}$$

So (37) implies that

$$D\varphi = F\varphi,$$

where $F^2 = \frac{1}{4}R + |Q^{\varphi}|^2$. Whereas equality case in (1) gives $\sqrt{R+4|Q^{\varphi}|^2} - |H|$ is constant, we cannot conclude here that $\frac{1}{4}R + |Q^{\varphi}|^2$ and H are constant as in the case of Zhang's inequality. Nevertheless, we have the following.

Corollary 4.2. *If H is constant, then equality case in* (1) *holds if and only φ is a T-Killing spinor*.

By hypothesis H has constant sign and we can conclude that λ has the same sign. Recall that p and q are related by

$$p = -\frac{1}{nq}$$

and

$$nq^2 = \frac{|H|}{\sqrt{R+4|Q^{\varphi}|^2} - |H|}.$$

Indeed, an easy calculation gives

$$0 = f = \left(p\,\frac{H}{2} + q\lambda\right) = \frac{(\mathrm{sign}(\lambda) - \mathrm{sign}(H))}{2\sqrt{n}}\sqrt{|H|(\sqrt{R+4|Q^{\varphi}|^2} - |H|)}. \tag{39}$$

Hence

$$\mathrm{sign}(\lambda) = \mathrm{sign}(H).$$

Remark 4.3. Equality case of (26) is included in that of (1): if we assume that φ is a Killing spinor, then necessarily $Q^{\varphi}_{ij} = (\lambda_1/n)\delta_{ij}$ and so $(\mathrm{tr}\, Q^{\varphi})^2 = \lambda_1^2 = \frac{1}{4}n/(n-1)R$. Therefore Eq. (38) implies

$$4|Q^{\varphi}|^2 = \frac{n}{n-1}R - R$$

and we have

$$\lambda^2 = \left(\sqrt{\frac{n}{n-1}R} - |H|\right)^2.$$

Remark 4.4. The previous remark shows that Theorem 1.1 improves Theorem 4.1. In particular, it does not require R to be positive, and the limiting case does not imply that H has to be constant.

5. Proof of the Theorem 1.2

Consider a conformal change of the metric $\bar{g} = \mathrm{e}^{2u}\widetilde{g}$ for any real function u on N. For simplicity, let $\bar{N} = (N, \bar{g})$. The natural isometry between $\mathrm{SO}\,N$ and $\mathrm{SO}\,\bar{N}$ induced by this conformal change of the metric lifts to an isometry between the $\mathrm{Spin}(n+1)$-principal bundles $\mathrm{Spin}\,N$ and $\mathrm{Spin}\,\bar{N}$, and hence between the two corresponding hypersurface spinor bundles S and $\bar{S}$. If $\varphi \in \Gamma(S)$, denote by $\bar{\varphi} \in \Gamma(\bar{S})$ its image by this isometry. Let $(\cdot,\cdot)_{\bar{g}}$ be the metric on $\bar{S}$ naturally defined as described in Section 2. Then for φ, ψ two sections of S, we have

$$(\varphi, \psi) = (\bar{\varphi}, \bar{\psi})_{\bar{g}}, \qquad \bar{X}\bar{\cdot}\bar{\psi} = \overline{X \cdot \psi}.$$

We will also denote by $\bar{g} = \mathrm{e}^{2u}_{|M} g$ the restriction of $\bar{g}$ to M. By conformal covariance of the Dirac operator, we have, for $\varphi \in \Gamma(S)$, (see [9])

$$\bar{D}(\mathrm{e}^{-((n-1)/2)u}\,\bar{\varphi}) = \mathrm{e}^{-((n+1)/2)u}\,\overline{D\varphi}, \tag{40}$$

where $\bar{D}$ stands for the Dirac operator w.r.t. $\bar{g}$. On the other hand

$$\bar{H} = \mathrm{e}^{-u}(H + n\,\mathrm{d}u(\nu)). \tag{41}$$

Therefore, if $\bar{D}_{\bar{H}}$ stands for the hypersurface Dirac operator w.r.t. $\bar{g}$, Eqs. (40) and (41) imply that,

$$\bar{D}_{\bar{H}}(\mathrm{e}^{-((n-1)/2)u}\,\bar{\varphi}) = \mathrm{e}^{-((n+1)/2)u}(\overline{D_H\varphi} - \tfrac{1}{2}n\,\mathrm{d}u(\nu)\bar{\varphi}). \tag{42}$$

Remark 5.1. We see that if $\mathrm{d}u(\nu)_{|M} = 0$, D_H is a conformal invariant operator. In this case, techniques used in [7] can be applied for the eigenvalues of D_H. Indeed, such a conformal change of metric can be viewed as a intrinsic conformal change of the metric on M, when we omit the ambient space N (See Section 7).

From now on, we will only consider conformal changes of the metric $\bar{g} = \mathrm{e}^{2u}\widetilde{g}$ with $\mathrm{d}u(\nu) = 0$ on M. They will be called regular conformal changes of metric as in [9].

For $\varphi \in \Gamma(S)$ an eigenspinor of D_H with eigenvalue λ, let $\bar{\psi} := \mathrm{e}^{-((n-1)/2)u}\,\bar{\varphi}$. Then (42) gives

$$\bar{D}_{\bar{H}}\,\bar{\psi} = \lambda_H\,\mathrm{e}^{-u}\,\bar{\psi}. \tag{43}$$

Recall that

$$\bar{\nabla}_i\bar{\varphi} = \overline{\nabla_i\varphi} - \tfrac{1}{2}\overline{e_i \cdot \mathrm{d}u \cdot \varphi} - \tfrac{1}{2}e_i(u)\bar{\varphi}, \tag{44}$$

and $\overline{e_i} = \mathrm{e}^{-u}e_i$. Now, as in [7], it is straightforward to get

$$\begin{aligned} \bar{Q}^{\bar{\psi}}_{\bar{i}\bar{j}} &= \tfrac{1}{2}(\overline{e_i}\bar{\cdot}\bar{\nu}\bar{\cdot}\bar{\nabla}_{\overline{e_j}}\bar{\psi} + \overline{e_j}\bar{\cdot}\bar{\nu}\bar{\cdot}\,\bar{\nabla}_{\overline{e_i}}\bar{\psi}, \bar{\psi}/|\bar{\psi}|^2_{\bar{g}})_{\bar{g}} \\ &= \tfrac{1}{2}\mathrm{e}^{-u}(\overline{e_i}\bar{\cdot}\bar{\nu}\bar{\cdot}\bar{\nabla}_{e_j}\bar{\varphi} + \overline{e_j}\bar{\cdot}\bar{\nu}\bar{\cdot}\bar{\nabla}_{e_i}\bar{\varphi}, \bar{\varphi}/|\bar{\varphi}|^2_{\bar{g}})_{\bar{g}} \\ &= \tfrac{1}{2}\mathrm{e}^{-u}(e_i \cdot \nu \cdot \nabla_{e_j}\varphi + e_j \cdot \nu \cdot \nabla_{e_i}\varphi, \varphi/|\varphi|^2) = \mathrm{e}^{-u}\,Q^{\varphi}_{ij}. \end{aligned} \tag{45}$$

Hence,

$$|\bar{Q}^{\bar{\psi}}|^2 = \mathrm{e}^{-2u}|Q^{\varphi}|^2 \tag{46}$$

Eq. (22), which is also true on N, applied to $\bar{\psi}$ yields

$$\int_M |\bar{\nabla}^Q \bar{\psi}|^2 v_{\bar{g}} = \int_M (1+nq^2)\left[\lambda^2\,\mathrm{e}^{-2u} - \frac{1}{4}\left(\frac{\bar{R}+4|\bar{Q}^{\bar{\psi}}|^2}{1+nq^2} - \frac{\bar{H}^2}{nq^2}\right)\right]|\bar{\varphi}|^2 v_{\bar{g}} \tag{47}$$

which, because of (41) and (46) gives

$$\int_M |\bar{\nabla}^Q \bar{\psi}|^2 v_{\bar{g}} = \int_M (1+nq^2)\left[\lambda^2 - \frac{1}{4}\left(\frac{\bar{R}\,\mathrm{e}^{2u}+4|Q^{\varphi}|^2}{1+nq^2} - \frac{H^2}{nq^2}\right)\right]\mathrm{e}^{-2u}|\bar{\varphi}|^2 v_{\bar{g}}. \tag{48}$$

Taking

$$nq^2 = \frac{|H|}{\sqrt{\bar{R}\,\mathrm{e}^{2u}+4|Q^{\varphi}|^2} - |H|}$$

completes the proof of Theorem 1.2.

6. General limiting cases

We now discuss the limiting case in inequality (2). Equality holds if and only if $\bar{\nabla}^Q_{\bar{i}}\bar{\psi} = 0$ for $1 \le i \le n$, which can be written as

$$0 = \bar{\nabla}_{\bar{i}}\bar{\psi} + \left(p\frac{\bar{H}}{2} + q\,\mathrm{e}^{-u}\lambda\right)\overline{e_i\cdot\nu\cdot}\bar{\psi} + \sum_j \bar{Q}^{\bar{\psi}}_{ij}\overline{e_j\cdot\nu\cdot}\bar{\psi}.$$

Since $\bar{\psi} := \mathrm{e}^{-((n-1)/2)u}\bar{\varphi}$, (44) and (46) yield

$$\begin{aligned} 0 &= \mathrm{e}^{-((n-1)/2)u}\,\mathrm{e}^{-u} \\ &\quad\times\left[\overline{\nabla_i\varphi} - \frac{1}{2}\overline{e_i\cdot \mathrm{d}u\cdot\varphi} - \frac{n}{2}e_i(u)\bar{\varphi} + \left(p\frac{H}{2}+q\lambda\right)\overline{e_i\cdot\nu\cdot\varphi} + \sum_j Q^{\varphi}_{ij}\overline{e_j\cdot\nu\cdot\varphi}\right]. \end{aligned} \tag{49}$$

With respect to the identification of Proposition 2.1, and by (14), this last statement is equivalent to

$$\nabla_i\varphi^* = \frac{1}{2}e_i\cdot \mathrm{d}u\cdot\varphi^* + \frac{n}{2}\,\mathrm{d}u(e_i)\varphi^* - fe_i\cdot\varphi^* - \sum_j Q^{\varphi}_{ij}e_j\cdot\varphi^*. \tag{50}$$

where $f := p(H/2) + q\lambda$. As in Section 4, let $T_{ij} = Q^{\varphi}_{ij} + f\delta_{ij}$. It is then straightforward to prove that $T_{ij} = Q^{\varphi}_{ij}$ and so $f = 0$.

Taking the scalar product of (50) with φ^*, it follows:

$$\begin{aligned} \tfrac{1}{2}e_i(|\varphi|^2) &= (\nabla_i\varphi^*, \varphi^*)_{\Sigma M} \\ &= \tfrac{1}{2}(e_i\cdot \mathrm{d}u\cdot\varphi^*, \varphi^*)_{\Sigma M} + \tfrac{1}{2}n\,\mathrm{d}u(e_i)|\varphi|^2 = \tfrac{1}{2}(n-1)\,\mathrm{d}u(e_i)|\varphi|^2. \end{aligned}$$

Therefore,

$$\mathrm{d}u = \frac{\mathrm{d}|\varphi|^2}{(n-1)|\varphi|^2}. \tag{51}$$

So we proved that equality holds in (2) if and only if the eigenspinor φ satisfies

$$\nabla_i \varphi^* = \frac{1}{2} e_i \cdot \mathrm{d}u \cdot \varphi^* + \frac{n}{2}\,\mathrm{d}u(e_i)\varphi^* - \sum_j Q^{\varphi}_{ij} e_j \cdot \varphi^*$$

with u satisfying (51). Such field equations have already been studied, as well as their integrability conditions, by Friedrich and Kim [5]. We will call them WEM-spinors (for weak energy–momentum spinors). If they satisfy the Einstein–Dirac equation, they are called WK-spinors (for weak Killing spinors). These are exactly the limiting case of Hijazi's equality involving conformal change of the metric and the energy–momentum tensor [8], in which case, they are also eigenspinors for the classical Dirac operator.

In our situation, there are not eigenspinors for D. As a consequence, even if in the limiting case $\sqrt{\bar{R}\,\mathrm{e}^{2u} + 4|Q^{\varphi}|^2} - |H|$ has to be constant, we cannot conclude that both $\sqrt{\bar{R}\,\mathrm{e}^{2u} + 4|Q^{\varphi}|^2}$ and H are constant.

Nevertheless, as in the previous section, a simple calculation leads to

$$0 = f = \mathrm{e}^u \frac{\mathrm{sign}(\lambda) - \mathrm{sign}(H)}{2\sqrt{n}} \sqrt{|H| \left(\sqrt{\bar{R}\,\mathrm{e}^{2u} + 4|Q^{\varphi}|^2} - |H|\right)}.$$

Hence

$$\mathrm{sign}(\lambda) = \mathrm{sign}(H).$$

Now recall the inequality proved by Hijazi and Zhang:

Theorem 6.1 (Hijazi and Zhang [9]). *Let $M^n \subset N^{n+1}$ be a compact hypersurface of a Riemannian spin manifold $(N, \widetilde{g})$. Assume that $n \geq 2$ and $n\bar{R}\,\mathrm{e}^{2u} > (n-1)H^2 > 0$ for some regular conformal change of the metric $\bar{g} = \mathrm{e}^{2u}\widetilde{g}$. Then if λ is any eigenvalue of the hypersurface Dirac operator D_H, one has*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left(\sqrt{\frac{n}{n-1}\bar{R}\,\mathrm{e}^{2u}} - |H|\right)^2. \tag{52}$$

As in the proof of Theorem 1.2, Theorem 6.1 is obtained by using the modified connection defined by (27), on the manifold $(N, \bar{g} = \mathrm{e}^{2u}\widetilde{g})$.

As in the beginning of this section, it is then easy to see that equality holds in (52) if and only if

$$0=\mathrm{e}^{-((n-1)/2)u}\,\mathrm{e}^{-u}[\overline{\nabla_i\varphi} - \tfrac{1}{2}\overline{e_i \cdot \mathrm{d}u \cdot \varphi} - \tfrac{1}{2} n e_i(u)\bar{\varphi} + (p(\tfrac{1}{2}H)+q\lambda)\overline{e_i \cdot \nu \cdot \varphi}]. \tag{53}$$

With respect to the identification of Proposition 2.1, and by (14), this last statement is equivalent to

$$\nabla_i \varphi^* = \tfrac{1}{2} e_i \cdot \mathrm{d}u \cdot \varphi^* + \tfrac{1}{2} n\,\mathrm{d}u(e_i)\varphi^* - f e_i \cdot \varphi^*. \tag{54}$$

where $f := p(H/2) + q\lambda$. As in Section 4, let $T_{ij} = f\delta_{ij}$. Then it is straightforward to prove that $T_{ij} = Q^{\varphi}_{ij}$ and that spinors fields satisfying the equality case in Theorem 6.1 are particular WEM-spinors. Now, by (38) and by (45), we see that necessarily

$$f = \pm\frac{1}{n}\sqrt{\frac{n}{n-1}\bar{R}\,\mathrm{e}^{2u}}. \tag{55}$$

Hence, solutions of (54) correspond exactly to sections verifying the limiting case of inequality (5.1) in [7].

Recall that here, p and q are given by

$$p = \frac{1-q}{1-nq}, \tag{56}$$

and

$$(1-nq)^2 = \frac{(n-1)|H|}{\sqrt{n/(n-1)\bar{R}\,\mathrm{e}^{2u}} - |H|}. \tag{57}$$

Therefore we can recover that $\mathrm{sign}(\lambda) = \mathrm{sign}(H)$ as made previously by computing explicitly $p(H/2) + q\lambda$. In fact, as in Remark 4.3, the consequence of Eq. (38) is that Theorem 1.2 improves Theorem 6.1.

7. Concluding remark

We conclude this paper by observing that all computations previously made could be done in an intrinsic way, considering a modified Dirac operator $D_f = D - \frac{1}{2}f$, and connections on ΣM:

$$\nabla^{\lambda}_i = \nabla_i + (p(f/2) + q\lambda)e_i$$

and

$$\nabla^{Q}_i = \nabla_i + (p(f/2) + q\lambda)e_i + Q^{\varphi}_{ij}e_j,$$

with an appropriate choice of p and q (simply replace H by f), in (28) and (30).

The identification of the spinor bundles of Section 2 allows to assert that computations will lead to the same results, but in a more general way. Therefore we can deduce the following proposition.

Proposition 7.1. *Let (M^n, g) be a compact Riemannian spin manifold. Assume that $n \geq 2$ and $nR > (n-1)f^2 > 0$, with $f : M \to \mathbb{R}$ a smooth function. Then for any eigenvalue λ of the Dirac–Schrödinger operator $D_f = D - \frac{1}{2}f$, one has*

$$\lambda^2 \geq \frac{1}{4}\inf_M\left(\sqrt{\frac{n}{n-1}R} - |f|\right)^2.$$

Equality hold if and only if M admits a Killing spinor and in this case (M^n, g) *is Einstein, f constant, and*

$$\mathrm{sign}(\lambda) = \mathrm{sign}(f).$$

Similarly, one obtain Proposition 7.2.

Proposition 7.2. *Let* (M^n, g) *be a compact Riemannian spin manifold. Let* λ *be any eigenvalue of the Dirac–Schrödinger operator* $D_f = D - \frac{1}{2}f$, *associated with the eigenspinor* φ. *Assume that* $R + 4|Q^\varphi|^2 > f^2 > 0$, *then*

$$\lambda^2 \geq \frac{1}{4}\inf_M \left(\sqrt{R + 4|Q^\varphi|^2} - |f|\right)^2,$$

where Q^φ *is the energy–momentum tensor associated with* φ.

If equality holds M admits an EM-spinor, and in this case,

$$\mathrm{sign}(\lambda) = \mathrm{sign}(f).$$

Now using a conformal change of the metric g, (see Remark 5.1), we prove Proposition 7.3 in the same way

Proposition 7.3. *Let* (M^n, g) *be a compact Riemannian spin manifold. Let* λ *be any eigenvalue of the Dirac–Schrödinger operator* $D_f = D - \frac{1}{2}f$, *associated with the eigenspinor* φ.

If $\bar{R}\,\mathrm{e}^{2u} + 4|Q^\varphi|^2 > f^2 > 0$, *where* $\bar{R}$ *is the scalar curvature of M for a conformal metric* $\bar{g} = \mathrm{e}^{2u} g$, *then*

$$\lambda^2 \geq \frac{1}{4}\inf_M \left(\sqrt{\bar{R}\,\mathrm{e}^{2u} + 4|Q^\varphi|^2} - |f|\right)^2.$$

Equality holds if and only if M admits a WEM-spinor, and in this case, the function u is uniquely defined up to a constant by

$$u = \frac{\ln(|\varphi|^2)}{(n-1)}.$$

Moreover,

$$\mathrm{sign}(\lambda) = \mathrm{sign}(f).$$

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